International Journal for Innovative Engineering and Management Research
A Peer Revieved Open Access International Journal
www.ijiemr.org

IJIEMR Transactions, online available on 16th Nov 2020. Link :http://www.ijiemr.org/downloads.php?vol=Volume-09&issue=ISSUE-12

DOI: 10.48047/IJIEMR/V09/I12/28

Title: **RADIALLY CRITICAL GRAPHS WITH GIVEN CYCLOMATIC NUMBER AND PENDANT VERTICES**

Volume 09, Issue 12, Pages: 173-177

Paper Authors

Yusuf Nishanov

International Journal for Innovative Engineering and Management Research
A Peer Revieved Open Access International Journal
www.ijiemr.org

UDC: 519.17

RADIALLY CRITICAL GRAPHS WITH GIVEN CYCLOMATIC NUMBER AND PENDANT VERTICES

Yusuf Nishanov
Associate Professor, Samarkand State Architecture and Construction Institute, Samarkand, Uzbekistan
E-mail: y.nishanov@umail.uz, tel.: +998(93)336-18-51

Abstract. In this paper, we study ordinary radially critical graphs without loops and multiple edges, with cyclomatic number $\lambda\leq 2$ and with pendant vertices $j\leq 2$.
Key words: Graph, vertex, edge, radius, pendant vertex, peripheral vertex, central block.

Introduction. Suppose we are given a graph $L=(X,U)$, where X is a set of vertices and U is a set of edges of this graph. Cyclomatic number of graph L is the number $\lambda=m-n+1\geq 0$, where m is the number of edges and n is the number of vertices of this graph. This means, there are such λ of edges, removal of which makes the graph into a spanning tree. Vertex x is called pendant in case its degree $s(x)=1$. Distance between vertices x and y is denoted by $\rho(x,y)$. Vertices x_1 and x_2 are called similar if

$$\{x\in X\setminus\{x_1\}/\rho(x_1,x)=1\}=\{x\in X\setminus\{x_2\}/\rho(x_2,x)=1\}.$$

Diameter of graph is $d(L)=\max_{x,y\in X}\rho(x,y)$.

Radius of graph is $r=\min_{x\in X}\left(\max_{y\in X}\rho(x,y)\right)$. A graph is called radially critical if, after adding any new edge, the graph radius decreases.

The author previously proved ([1]) that radially critical graphs can be obtained by expanding non-peripheral vertices of radially critical graphs without non-coincident similar vertices.

Cartesian product of graphs $L=(X, U)$ and $G=(Y, V)$ is a graph $T=(Z, W)=L\times G$, where $X=(x_1,x_2,\dots,x_n)$, $Y=(y_1,y_2,\dots,y_p)$, $U=(u_1,u_2,\dots,u_m)$, $V=(v_1,v_2,\dots,v_q)$, $Z=\{(x_i,y_j)/x_i\in X, y_j\in Y\}$, $W=\{(x_iy_j,x_ky_l)\}$; at that the edges $w=\{(x_iy_j,x_ky_l)\}$ of obtained graph exist only for those pairs of vertices, when either $x_i=x_k\in X$ and $\exists y_jy_l\in V$, or $y_j=y_l\in Y$ and $\exists x_ix_k\in U$.

Lemma 1. If in a radially critical graph L, the vertex x_0 is a peripheral vertex, then there is central vertex z_0 such that $\rho(z_0,x_0)=r$, $|x\in L\setminus max_{x\in L}\,\rho(z_0,x)=r|=1$.

Lemma 2. If in a radially critical graph L, the vertex x_0 is a peripheral vertex, then any central vertex z_0 of graph L_{+u}, where $u=yy'$, satisfying the conditions $\rho(z_0,x_0)=\rho(z_0,y)+\rho(y,y')+\rho(y',x_0)=r$, $\rho(z_0,y)=r-3$, $\rho(y,y')=2$, $\rho(y',x_0)=1$, is also a central vertex for the source graph L.

Lemma 3. If for central vertex y_0 in a radially critical graph L, we have $|y\in L\setminus max_{x\in L}\,\rho(y_0,x)=r|>1$, where y is pendant, then the vertex y_0 is not central for L_{+u}, where $u=yy'$, $\rho(y_0, y')=\rho(y_0, y)=r-2$, $\rho(y, y')=2$. These Lemmas are used to prove the following theorem.

Theorem 1. *Graph $T=L\times G$ is radially critical if and only if one of the graphs is either a radially critical graph without non-*

International Journal for Innovative Engineering and Management Research
A Peer Revieved Open Access International Journal
www.ijiemr.org

coincident similar vertices, where all the vertices are central, or graph F_2, i.e. complete double hump graph; and the other one is an arbitrary radially critical graph without non-coincident similar vertices.

Proof. Sufficiency is quite obvious, therefore we prove only necessity. We assume that L and G are arbitrary radially critical graphs without non-coincident similar vertices, where the radius of graph L is equal to r_1, and the radius of graph G is equal to r_2. Accordingly, the radius of obtained graph T is equal to r_1+r_2. Without loss of generality, it can be considered that for L and G the vertices x_0 and $\bar{x}_0$ are respectively peripheral, while z_0 and $\bar{z}_0$ are central vertices of these graphs, satisfying the conditions of Lemmas 1 and 2. In such a case, the vertex $(x_0\bar{x}_0)\in T$ is peripheral.

Let us assume that in graph L we have $\rho(z_0,x_0)=\rho(z_0,y)+\rho(z_0,y)+\rho(y,y')+ +\rho(y',x_0)=r$, $\rho(z_0,y)=r-2$, $\rho(y,y')=\rho(y',x_0)=1$, and in graph G, in view of the fact that $\rho(\bar{z}_0,t)=r-2$, $\rho(t,t')=\rho(t',x_0)=1$, we have $\rho(\bar{z}_0,x_0)=\rho(\bar{z}_0,t)+ +\rho(t,t')+\rho(t',x_0)=r$.

Then, after adding new edge $w=(yt,y't')$ to the graph T, the radius of resulting graph T_{+w} remains equal to r_1+r_2, i.e., it does not decrease, since peripheral vertex $(x_0\bar{x}_0)$ remains at distance r_1+r_2 from all central vertices of resulting graph (because it follows from Lemmas 1 and 2 that they all are central also for source graph T). Consequently, graph T is not a radially critical graph. The theorem is proved.

Let $G(j, \lambda)$ denote the class of radially critical graphs having a cyclomatic number λ and per j of pendant vertices.

It is known that with $\lambda=0$ each graph is a tree, and therefore class $G(j,0)$ is empty.

Class $G(j,1)$ is fully described in [2], moreover, it has been proved that $4\leq j\leq \frac{l}{2}$, where l is the length of a single cycle, and l is even-numbered.

Classes $G(j, 2)$, when $0\leq j\leq 2$, are described here. Since it is sufficient to study radially critical graphs without non-coincident vertices, we can assume that all cycles of the length $l\geq 3$ are in central block ([3]). Without loss of generality, it can be considered that central block is comprised of three intercrossing simple chains P_1, P_2 and P_3 with common endpoints y_1 and y_2, where $l(P_1)\geq l(P_2)\geq l(P_3)$.

Results.

Proposition 1. Class G(0,2) is empty.

Proof. If $l(P_1\cup P_3)\leq 2r-1$, then the radius of graph is less than r, since $\max_x \rho(y_1, x)\leq r-1$. Therefore, $l(P_1\cup P_3)\geq 2r$. Then, to decrease radius in the source graph with $l(P_2)>2$ and $l(P_3)>1$, adding of the edge $u=x_2x_3$, where $x_2\in P_2$, $x_3\in P_3$, $\rho(x_2,y_2)=\rho(x_3,y_2)=1$, and adding of the edge $u'=x'_2x'_3$, where $x'_2\in P_2$, $x'_3\in P_3$, $\rho(x'_2,y_1)=\rho(x'_3,y_1)=1$, requires existence of at least two completely different central vertices. Let these central vertices be z_0 and z'_0. Then, due to the fact that $l_3\leq l_2$, the outermost points from these central vertices are in chain P_2. In such a situation, there would be $l(P_1\cup P_2)< r-1$, which is impossible, since then there would be a vertex $\bar{z}\in P_1\cup P_2$ such that $\max_x \rho(\bar{z},x)=r-1$.

If $l(P_3)=1$, then the point y_1 can be taken in place of the point x_3, and the point y_2 can be taken in place of the point x'_3, the result will be exactly the same.

In the case when $l(P_2)=2$ and $l(P_3)=1$, there should be $l(P_1\cup P_2)\leq 2r-1$. If not, adding of the edge $u=x_2z_0$, where $\rho(x_2,z_0)=2$, $z_0\in P_1$, (or the edge $u=x'_2z'_0$, where $\rho(x'_2,z'_0)=2$, $z'_0\in P_1$) does not

International Journal for Innovative Engineering and Management Research
A Peer Revieved Open Access International Journal
www.ijiemr.org

decrease the radius of graph, since in both cases

$\forall x_0 \in G\ \exists x'_0 \in P_1 \cup P_3$ $[\rho_{G+u}(x_0, x'_0) \geq r]$, which contradicts the criticality of the graph. Consequently, the source graph is not a radially critical and the class $G(0, 2)$ is empty.

Proposition 2. Class $G(1,2)$ is empty.

Proof. Then, in such graph there is only one cutpoint with hanging pendant chain. Let y_3 be a cutpoint, and P_0 be a pendant chain of the lengh k, where $\bar{x}$ be a pendant vertex of this chain.

1. $y_3 \in P_1 \setminus \{y_1, y_2\}$. In view of the fact that the graph diameter is no more than $2r$-2, $\exists x \in P_1 \cup P_2 \cup P_3[\max_x \rho(\bar{x}, x) \leq 2r-2]$ and $l(P_1 \cup P_2) \leq 2r-k$. If here $l(P_1 \cup P_2)=l(P_1)+ l(P_2)=l_1+l_2 \geq 2r-k$, then adding of the edge $u=y'_3y''_3$, where $y'_3 \in P_1$, $y''_3 \in P_0$, $\rho(y''_3,y_3)=\rho(y_3,y'_3)=1$, does not decrease the radius of graph, since $\exists x \in P_2[\rho_{G+u}(z_0, x) \geq r]$ occurs for any central vertex z_0 of source graph, which is impossible due to the criticality of graph. Consequently, $l_1+l_2 \leq 2r-k-1$. Assume that $l_2= l_3$. It is obvious that $l_2= l_3 \geq 2$. In this case adding of the edge $u=y'_1\ y''_1$, where $\rho(y'_1,y''_1)=2$, $\rho(y''_1,y_1)=\rho(y_1,y'_1)=1$, $y'_1 \in P_2$, $y''_1 \in P_3$ does not decrease the radius of graph, since $l_2+l_3 < 2r-k-1$. Consequently, $l_2 > l_3$.

Let us prove that $\rho(y_3,y_1)=\rho(y_3,y_2)$. Indeed, if $l_1 \geq 5$, then adding of edges of type $u=y'_3y''_3$ or $u=\bar{y}'_3y''_3$, where $y'_3 \in P_1$, $\bar{y}'_3 \in P_1$, $\rho(y_3,\bar{y}'_3)=1$, $\rho(y'_3,\bar{y}'_3)=2$, shows that there exist z_0 and z'_0 central vertices, for which $\rho(z_0,\bar{x})= \rho(z'_0,\bar{x})=r$. Then $\exists \bar{x}' \in P_2[\rho(z_0,\bar{x}')=\max_{x \in Q} \rho(z_0,x)=r\text{-}1]$ and $\exists\ \bar{x}'' \in P_2[\rho(z'_0,\bar{x}'')=\max_{x \in Q} \rho(z'_0,x)=r\text{-}1]$, where $Q= P_2 \cup P_3$.

Therefore $l_1+l_2=2r$; otherwise, adding of the edge $u=y'_3\bar{y}'_3$ does not decrease the radius of graph. In this case, all vertices from z_0 to z'_0 in P_1 chain will be central (odd number and not less than five vertices). From here it follows that l_1 is even-numbered. Note that $l_1<5$ is impossible.

Assume $l_3=1$. Let us add edge $u=y_2y'_2$, to the source graph, where $\rho(y_1,y'_2)=2$, $\rho(y_2,y'_2)=1$, $y'_2 \in P_2$ or $u=y_2y'_1$, where $\rho(y_2,y'_1)=2$, $\rho(y_1,y'_1)=1$, $y'_1 \in P_2$. Accordingly, it is obvious that l_2+l_3 is an even number, and l_2 is an odd number. In this case, adding of the edge $u=y_3y'_3$ does not decrease the radius of graph. Consequently, $l_3 \geq 2$. Since l_1 is even-numbered and $l_1+l_2=2r$, l_2 is also be even-numbered; therefore, l_3 will be an even number. That way, if $l_2= l_3+2$, then adding the edge of type $u=y''_1\bar{y}'_1$, where $\rho(y'_1,\bar{y}'_1)=1$, $\rho(y_1,\bar{y}'_1)=2$, $\bar{y}'_1 \in P_2$, does not decrease the radius of graph. Therefore, under these conditions, we would have $l_1 \geq l_2 \geq l_3+4$, which is impossible.

2. $y_3 \in P_2 \setminus \{y_1, y_2\}$. Similarly to item 1, it is proved that $\rho(y_1,y_3)=\rho(y_3,y_2)$ and l_1, l_2, l_3 are even numbers, $l_1 \geq l_2 \geq l_3+2$. It is known that with $l_2=4$, we would have $l_1=6,8,10$, etc. Consequently, this case is also impossible.

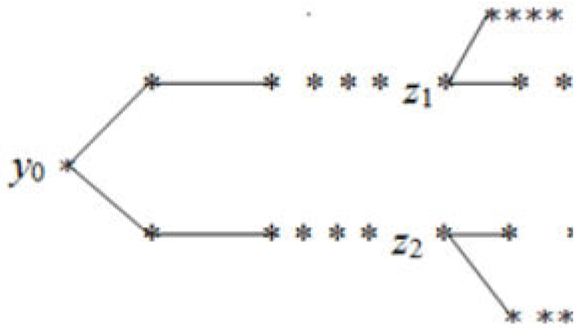
3. $y_3 \in P_3 \setminus \{y_1, y_2\}$. This case is also impossible, since adding of the edge $u=y'_1y''_1$ (or $u= y'_2y''_2$) does not decrease the radius of graph.

4. $y_3 \in \{y_1, y_2\}$. Without loss of generality, it can be considered that $y_3=y_1$. Assume that after adding the edge $u=c_1y'_1$, where $c_1 \in P_0$, $y'_1 \in P_1$, $\rho(c_1,y'_1)=2$, the radius of graph decreases by one unit. Moreover, ether a single vertex $z_1 \in P_1$, where $\rho_{G+u}(z_1,\bar{x})=r-1$, or the vertex $\bar{z}$ from $P_2 \cup P_3$, where $\rho_{G+u}(\bar{z},\bar{x})=r-1$, is a central vertex in the

obtained graph. Then adding of the edge $u=x_2x_3$, where $x_2\in P_2$, $x_3\in P_3$, $\rho(x_2,x_3) = 2$, $\rho(x_2, y_3) = \rho(y_3, x_3) = 1$, should decrease the radius of source graph by a unit. In the obtained graph $G+u$, $\overline{z}'$ will be the central vertex for which $\rho(\overline{z}',y_2)+ \rho(y_2, x_3)+1+ \rho(x_2,x'_2)=r-1$, where x'_2 is the most distant point from $\overline{z}'$ in the source graph. Consequently, it would be central vertex for the source graph. In such a case, $\max(l(P_1\cup P_3), l(P_1\cup P_2))=r$, otherwise, adding of edges $u=y'_1x_2$ and $u= y'_1x_3$ does not decrease the radius of graph, which is impossible due to the criticality of graph. It follows that this case is also impossible.

Consequently, class $G(1, 2)$ is empty.

Theorem 2. *For any odd $r\geq 3$ there is graph $L\in G(2,2)$, having $2r+2[\frac{r-1}{2}]$ vertices, two pendant chains of length $k=[\frac{r-1}{2}]$ and $2r+2[\frac{r-1}{2}]+1$ edges (Fig.1.)*



Proof. Let $r\geq 3$ be odd-numbered. Assume that z_1 and z_2 are cutpoints, where P_4 and P_5 are pendant chains of graph, x_1 and x_2 are respectively their pendant vertices. Let $l_2=2$ and $l_3=1$.

1. We first check radially critical graphs with radius $r(L)=3$. Then, due to the existence of pendant vertices, the graph diameter cannot be equal to 3, therefore $d(L)=4$, $\rho(z_1, z_2)=1$.

Assume that there is central vertex y_0 such that $\rho(y_0, z_2)= =\rho(y_0, z_2)=2$, $\max_x \rho(y_0,x)= \rho(y_0,x_1)= \rho(y_0,x_2)=3$. In such a case, according Lemma 3, this vertex is not central for L_{+u}, where $u=yy'$ is any new added edge.

Let $|\{x\backslash\max\rho(y_0,x)=3\}|>1$. Assume that $\{x\backslash\max\rho(y_0,x)=3\}=\{\bar{x},\bar{x}'\}$, and add edges $u=z_{10}\bar{x}$ to graph L, where $\rho(y_0,z_{10})=1$, $\rho(z_{10},\bar{x})=2$. It is clear, if y_0 is in chain l_1, then vertex $\bar{x}$ is either in chain l_2, or it is pendant. If $\bar{x}\in l_2$, then $l_2>2$, and we have $\rho_{G+u}(y_0,\bar{x}_1)=3$ for vertex $\bar{x}_1\in l_2$, which is impossible due to criticality of the graph. If vertex $\bar{x}$ is pendant, then we again have $\rho_{G+u}(y_0,\bar{x})=3$. Consequently, $|\{x\backslash\max\rho(y_0,x)=3\}|=1$. In this case, it is obvious that $l_1=5$. Among other things, vertices x_1 and y_0 are adjacent to vertex z_1, while vertices x_2 and y_0 are adjacent to vertex z_2, $\rho(y_3,\bar{x})= \rho(y_4,\bar{x})=1$, i.e. the number of vertices of such graph $n(L)=8$, and the number of edges $m(L)=9$.

2. Let us see now the case, when radius of graph $r(L)\geq 4$. Let $l(P_4)=k$.

If the chain is hung on to vertex y_1, then adding of edge $u=y_{11}z$, where $\rho(y_{11},z)=2$, $z\in P_2/\{y_1,y_2\}$, $\rho(y_1,y_2)=1$, and $y_{11}\in P_4$, does not decrease the radius of graph. Accordingly, one of the chains may be hung on only to vertex z from vertices $P_2\cup P_3$. We would have $\rho(x_{01},z)= =\rho(x_{01},t_1)+\rho(t_1,y_2)+\rho(y_2,z)=r-k$ for vertex $x_{01}\in P_1$ in graph L . Then adding of edge $u=t_1z$, where $t_1\in P_1$, $\rho(t_1,y_1)=2$, $\rho(t_1,y_2)=1$ and $\rho(t_1,z)=2$, does not decrease the radius of graph, since $\rho_{Lu}(x_{01},z)= \rho_{Lu}(x_{01},t_1)+\rho_{Lu}(t_1,y_1)+ \rho_{Lu}(y_1,z)=r-k$. Consequently, it is also impossible to hang on simple chain to vertex z. Therefore both chains will be hung on to the chain $P_1\backslash\{y_1,y_2\}$. Due to the simplicity of the graph, it is clear that it is located symmetrically with respect to the vertices z and y_0, where $\rho(y_0,z)=r$. Obviously, these chains will be hung closer to the vertex y_0, than to the vertex z. Let x_1 and x_2 be pendant vertices, z_1 and z_2 be cutpoints, $\rho(z_1,x_1)=\rho(z_2,x_2)=k$, $\rho(y_0,z_1)=\rho(y_0,z_2)=r-k-1$. Then $\rho(y_0,x_1)=\rho(y_0,x_2)= =r-1$, $\rho(x_1,x_2)=\rho(x_1,z_1)+\rho(z_1,y_0)+\rho(y_0,z_2)+\rho(z_2,x_2) =2r-2$, $\rho(y_1,z_1)=\rho(y_2,z_2)=k-1$. Now, if we add edges $u=y_2y_{10}$, where $\rho(y_1,y_{10})=1$, $\rho(y_2,y_{10})=2$, then vertex y_2 will be central vertex for graph L_u. Therefore, $\rho(x_1,z_1)+(\rho(z_1,y_1)+1+\rho(y_2,z_2))+\rho(z_2,x_2)=k+(l_1+l_3-2k)+k=2k+((2r-1)-2k)=2r-1$. On the other hand, $\rho(x_1,z_1)+(\rho(z_1,y_1)+1+\rho(y_2,z_2))+\rho(z_2,x_2)=k+(k+1+k)+k=4k+1$, from which $2r-1=4k+1$, or $r=2k+1$, i.e. r is an odd number and $k=[\frac{r-1}{2}]\in N$.

Conclusion. Thus, for any odd $r\geq 3$ there is graph $L\in G(2,2)$ with $2r+2[\frac{r-1}{2}]$ vertices, two pendant chains of the length $k=[\frac{r-1}{2}]$ and $2r+2[\frac{r-1}{2}]+1$ edges.

References

1. Нишанов Ю. Радиально - критические графы с максимальным диаметром. “Вопросы вычислительной и прикладной математики”, 1972 г., вып. 15, Ташкент.
2. Нишанов Ю. Моноциклические радиально-критические графы. “Вопросы алгебры и теории чисел”, Самарканд.: изд. СамГУ, 1990 г., 54-65.
3. Зыков А.А. Основы теории графов. – М.: “Наука”, 1987 г., 384 с.