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# A CRITICAL STUDY ON SPECTRAL, K-SPECTRAL THEORY AND INEQUALITIES OF K-IDEMPOTENT MATRICES

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**ABSTRACT**
This abstract provides an overview of three interconnected topics in matrix theory: spectral theory, K-spectral theory, and inequalities of K-idempotent matrices. Spectral theory is a fundamental branch of linear algebra that deals with the study of eigenvalues and eigenvectors of matrices. It plays a crucial role in various applications across different fields, including physics, engineering, and data analysis. The extension of spectral theory to K-spectral theory involves the investigation of matrices with a specific class of eigenvalues known as K-eigenvalues. These K-eigenvalues are more general than the traditional eigenvalues, as they consider the properties of K-idempotent matrices. A K-idempotent matrix is a square matrix that satisfies the equation A^2 = KA, where K is a non-negative integer.

**Keywords: -** Special, Matrices, Eigenvalues, Matrices, Theory.

## I. INTRODUCTION

### k-EIGEN VALUE OF A MATRIX

In this section, we shall define a k-eigen value of a matrix as a special case of generalized eigen value problem $A\mathcal{X} = \lambda B$ for some matrices A and B. For that, first we define a permutation function $K(x)$ on the unitary space $\mathbb{C}^n$.

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n$ then K(x) is defined by,

$k(x) = \begin{pmatrix} x_{k(1)} \\ x_{k(1)} \\ \vdots \\ x_{k(1)} \end{pmatrix} \in \mathbb{C}^n$, where k is the fixed disjoint product of transpositions in $s_n$.

If K is the associated permutation matrix of k then it can be easily seen that $k(x) = kx$.

It is also clear that $K\left[K^{(x)}\right] = x$.

$$\text{i.e., } K\,[k(x)] = \begin{pmatrix} x_{k^2(1)} \\ x_{k^2(2)} \\ \vdots \\ x_{k^2(n)} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x \in \mathbb{C}^n$$

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**Example**

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \epsilon \mathbb{C}^4$ and let $k = \langle 1,4 \rangle \langle 2,3 \rangle$

Then $\mathcal{K}(x) = \begin{pmatrix} x_{k(1)} \\ x_{k(2)} \\ x_{k(3)} \\ x_{k(4)} \end{pmatrix} = \begin{pmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = kx$

Also $\mathcal{K}[\mathcal{K}(x)] = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x$

**Definition**

A k-eigen value of a matrix A is defined to be a zero of the polynomial det $(\lambda K - A) = 0$This polynomial is known as k-characteristic polynomial.

**Definition**

A non-zero vector $(x \neq 0)$ in $\mathbb{C}^n$ is said to be a k-eigen vector of a complex matrix A associated with a k-eigen value $\lambda$ if it satisfies $Ax = \lambda K^{(x)}$ where $K^{(x)}$ is as defined before. This is equivalent to $Ax = \lambda Kx$.

**Example**

$A = \begin{pmatrix} -1 & -1 & i \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a $\langle 1,2 \rangle$-idempotent matrix. The $\langle 1,2 \rangle$-eigen values of A are 1,1 and -1. A $\langle 1,2 \rangle$-eigen vector corresponding to the $\langle 1,2 \rangle$-eigen value 1 is $\begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}$It can be verified that $Ax = \lambda Kx$.

i.e., $\begin{pmatrix} -1 & -1 & i \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}$

**Theorem**

If A is a complex matrix in$\mathbb{C}^{n \times n}$ then

i. $(\lambda, x)$is a (k-eigen value, k-eigen vector) pair for A if and only if it is an (eigen value, eigen vector) pair for KA.
ii. Every matrix A satisfies the k- characteristic equation of KA.
iii. Any set of k-eigen vectors corresponding to distinct k-eigen values of a matrix must be linearly independent.

**Proof**

i. If $(\lambda, x)$ is a (*k*-eigen value,*k*-eigen vector) pair for *A* then

$$Ax = \lambda \mathcal{K}(x)$$
$$Ax = \lambda Kx$$

$$\mathrm{KAx} = \lambda \mathrm{x}$$

Therefore $(\lambda, \mathrm{x})$ is a (Eigen value, Eigen vector) pair for KA. By retracing the above arguments, we see that the converse is also true.

ii. By Cayley-Hamilton theorem, every matrix $A$ satisfies its characteristic equation. That is,

$\det(\lambda \mathrm{I} - \mathrm{A}) = 0$ (3.1)

$\det(\lambda \mathrm{K} - \mathrm{KA}) = \det[\mathrm{K}((\lambda \mathrm{I} - \mathrm{A})$

$= \det(\mathrm{K}) \det(\lambda \mathrm{I} - \mathrm{A}) = 0$

$= 0$ [by (3.1)]

Therefore the matrix A satisfies the k-characteristic equation of KA. In a similar manner, it can be easily proved that the matrix KA satisfies the k –characteristic equation of A.

iii. Any set of k-eigen vectors corresponding to distinct k-eigen values of a matrix $A$ is the set of eigen vectors corresponding to distinct eigen values of the matrix KA by what we have proved above in (i). Hence they are linearly independent.

## II. SPECTRAL CHARACTERIZATIONS OF $k$-IDEMPOTENT MATRICES

In this section, the spectral resolution of a $k$-idempotent matrix is determined as well as the diagonalizability of $k$-idempotent matrices is proved.

**Theorem**

Let A be a $k$-idempotent matrix. Then the eigen values of A are zero or cube root of unity.

**Proof**

Let λ be an eigen value of a $k$-idempotent matrixA. Then

$\mathrm{Ax} = \lambda \mathrm{x}$ (3.3)

$\mathrm{A}^2\mathrm{x} = \lambda \mathrm{Ax}$

$\mathrm{A}^2\mathrm{x} = \lambda^2 \mathrm{x}$ [by (3.3)] (3.4)

$\mathrm{A}^4\mathrm{x} = \lambda^4 \mathrm{A}^2\mathrm{x}$

$\mathrm{Ax} = \lambda^4 \mathrm{x}$

$\lambda \mathrm{x} = \lambda^4 \mathrm{x}$

$(\lambda^4 - \lambda)\mathrm{x} = 0$

Since $\mathrm{x} \neq 0$, we have $\lambda = 0$ or $1$, $\omega$ and $\omega^2$ where $\omega = \exp\left(\frac{2\pi i}{3}\right)$

**Example**

$\mathrm{A} = \begin{pmatrix} 1 & 2i & -i & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2i & i & 1 \end{pmatrix}$ is a $\langle 1,4\rangle\langle 2,3\rangle$-idempotent.

The eigen values of A are $\lambda = 1, 1, \omega, \omega^2$.

**Theorem**

If a matrix $\mathrm{A} \in \mathbb{C}^{n\times n}$ is $k$-idempotent then it is diagonalizable and the spectrum $\sigma(\mathrm{A}) \subseteq \{0, \omega, \omega^2, 1\}$ where $\omega = \exp\left(\frac{2\pi i}{3}\right)$. Moreover, there exist unique disjoint oblique projectors $\mathrm{P_i}$ for $i \in \{0,1,2,3\}$ such that

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$A = \sum_{j=1}^{3} \omega^j P_j$ (3.5)

$I = \sum_{i=0}^{3} P_i$ (3.6)

**Proof**

Since $A^4 = A$, the polynomial $q(t) = t^4 - t$ is a multiple of $q_A(t)$ of A and then every root of $q_A(t)$ has multiplicity 1. Hence the matrix A is diagonalizable.

Moreover, it is clear that $\sigma(A) \subseteq \{0, \omega, \omega^2, 1\}$ [by theorem 3.2.1]

We define $P_i$'s by the following formula,

$P_0 = \frac{f_0(A)}{f_0(0)}$, where $f_0(\lambda) = \prod_{i=1}^{3}(\lambda - \omega^i)$ and

$P_j = \frac{f_j(A)}{f_j(\omega^j)}$, where $f_j(\lambda) = \prod_{\substack{i=1 \\ i\neq j}}^{3} \lambda(\lambda - \omega^i)$ for j = 1,2,3

Using $1 + \omega + \omega^2 = 0$, we have

$P_0 = I - A^3$ : $P_1 = \frac{1}{3}(A^3 + \omega A^2 + \omega^2 A)$

$P_2 = \frac{1}{3}(A^3 + \omega^2 A^2 + \omega A)$ : $P_3 = \frac{1}{3}(A^3 + A^2 + A)$

In the case that $\omega^j \notin \sigma(A)$ for $j \in \{1,2,3\}$, we see that $P_j = 0$. Similarly $P_0 = 0$ when $0 \notin \sigma(A)$.

By spectral theorem, we see that the non-zero $P_i'$ s so obtained are disjoint oblique projectors (i.e., $P_i^2 = P_i$ and $P_i P_j = 0$ for $i \neq j$) to satisfy the decompositions (3.5) and (3.6). The uniqueness of the decompositions can be proved as follows:

Suppose if possible, let $Q_i'$ s be non-zero disjoint oblique projectors such that $A = \sum_{i=1}^{m} \alpha_i Q_i$, for complex numbers $\alpha_i$ and $I = \sum_{i=1}^{m} Q_i$. We wish to prove that this is actually identical with (3.5) and (3.6) except for notations and order of terms. First, it is proved that $\alpha_i$'s are precisely the eigen values of the matrix A.

Since $Q_i \neq 0$, there exists a non-zero vector x in the range of $Q_i$ such that $Q_i x = x$ and $Q_j x = 0$ for $j \neq i$.

$$Ax = \left(\sum_{i=1}^{m} \alpha_i Q_i\right) x$$

$$Ax = \alpha_i x$$

Therefore $\alpha_i$ is an eigen value of $A[\text{i.e.}, \alpha_i \in \{0, \omega, \omega^2, 1\}]$. Conversely, if $\lambda$ is an eigen value of A then $Ax = \lambda x$

$$\left(\sum_{i=1}^{m} \alpha_i Q_i\right) x = \lambda I x = \lambda \left(\sum_{i=1}^{m} Q_i\right) x$$

$$\sum_{i=0}^{m} (\lambda - \alpha_i) Q_i x = 0 \qquad (3.7)$$

Since $Q_i$ 's are disjoint, we can find at least one $x \neq 0$ among the non-zero vectors for which (3.7) is linearly independent. Hence, it follows that $\lambda = \alpha_i$ for some i. These arguments show that

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the set of $\alpha_i$ 's equals the set of eigen values of A. By suitably changing the order of terms, we have $A = \sum_{i=1}^{3} \omega^i Q_i$.
Since the expression for $P_i$ is unique in terms of A, we have $Q_i = P_i$ for $i \in \{0,1,2,3\}$. Hence the decompositions (3.5) and (3.6) are unique.

## III. *k*-SPECTRAL CHARACTERIZATIONS OF K-IDEMPOTENT MATRICES

A k-eigen value of a k-idempotent matrix is found and an equivalent condition for normality of a k-idempotent matrix is also determined in this section.

**Theorem**

Let A be a k-idempotent matrix. Then the k-eigen values of a are 0,1 and -1.

**Proof**

$$Ax = \lambda Kx \quad (3.10)$$
$$KAx = \lambda x \quad (3.11)$$

Pre multiplying by KA, we have

$$A^3x = \lambda KAx$$
$$A^3x = \lambda^2 x \quad \text{[using (3.10)]}$$

Pre multiplying by *A*, we have

$$Ax = \lambda^2 Ax$$
$$\lambda Kx = \lambda^3 Kx \quad \text{[using (3.10)]}$$
$$(\lambda - \lambda^3)Kx = 0$$

Since $Kx \neq 0$, we have $\lambda = 0,1,-1$.

**Example**

The $\langle 1,4\rangle\langle 2,3\rangle$-eigen values of example 3.2.2 are 1,1,-1,-1.

**Theorem**

If a matrix $A \in \mathbb{C}^{n\times n}$ is $k$-idempotent then $\sigma_k(A) \subseteq \{0,1,-1\}$. Moreover, there exist unique disjoint oblique projectors $Q_j$ for $j \in \{0,1,-1\}$ such that

$$KA = Q_1 - Q_{-1} \quad (3.12)$$
$$I = Q_0 + Q_1 + Q_{-1} \quad (3.13)$$

**Proof**

By (3.2) and theorem 3.3.1, it is clear that $\sigma_k(A) = \sigma(KA) \subseteq \{0,1,-1\}$.
We define $Q_j$'s by the following formula

$$Q_j = \prod_{\substack{i=0,1,-1 \\ i\neq j}}^{n} \frac{KA - iI}{j - i} \text{for } j = 0,1,-1$$

Then $Q_0 = I - A^3$ : $Q_1 = \frac{1}{2}(A^3 + KA)$ : $Q_{-1} = \frac{1}{2}(A^3 - KA)$

In the case that $j \notin \sigma(KA)$ for $j \in \{0,1,-1\}$, we have $Q_j = 0$. It can be proved that the non-zero $Q_j$'s so obtained are unique disjoint oblique projectors such that satisfying the decompositions (3.12) and (3.13) analogous to the proof of theorem 3.2.3.

**Example**

Consider the ⟨1,4⟩⟨2,3⟩-idempotent matrix A. The oblique projectors of KA are found to be

$$Q_0 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad Q_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1+\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & -1 & \frac{1-\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad Q_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1-\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1+\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It can be easily verified that the above projectors are disjoint and satisfy the decompositions (3.12)and (3.13).

## IV. CONCLUSION

Further it is proved that $k$-idempotent matrices are $\{3\}$-group periodic. A set of necessary and sufficient conditions for a linear combinations $C = c_1A + c_2B$ of two commutative idempotent matrices $A$ and $B$ to be $k$-idempotent, is listed analogous to theorem. Then it is generalized to the problem of characterizing all situations in which the linear combination $C = c_1A + c_2B$ (where $A$ is an idempotent matrix and $B$ is a tripotent matrix) to be $k$-idempotent, is thoroughly studied analogous.

Various generalized inverses of a $k$-idempotent matrix are studied and the corresponding inverses for the elements in group $G = \{A, A^2, A^3, KA, AK, KA^3\}$ are determined. A condition for the Moore Penrose inverse of a $k$-idempotent matrix to be $k$-idempotent is derived. A column and row inverse of a $k$-idempotent matrix is found and then it is shown that the group inverse of a $k$-idempotent matrix $A$ is $A^2$. A commuting pseudo inverse of the corresponding elements in group $G$ is also found. The $k$-idempotency of EP matrices is analyzed in this chapter. An equivalent condition for a $k$-idempotent matrix to be EP is also determined.

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